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New Modal Synthesis Technique Using Mixed Modes

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A new modal synthesis technique using mixed modes is introduced. The method not only has a simple form, high precision, and efficiency, but it also provides two reliable criteria for determining the truncation frequency of system synthesis modes. These have been demonstrated in detail by the numerical examples. The key point of the method is as follows. The exact residual modes containing the effects of higher-order, free-interface modes are expressed analytically in terms of some lower-order, fixed-interface modes, and then the substructural displacements are expressed accurately in terms of some lower-order, mixed modes, i.e., the linear combinations of the lower-order, fixed-interface modes and lower-order, free-interface modes, instead of the assumed substructural displacements employed in general modal synthesis, so that only linear synthesis equations are involved in this new method, although the contributions of all higher-order modes are included.

Nomenclature

f	= frequency
$\boldsymbol{f}$	= force vector
H	= number of higher-order modes
$\boldsymbol{I}$	= identity matrix
j	= number of interface degrees of freedom (DOF)
$\boldsymbol{K}, \boldsymbol{k}, \bar{\boldsymbol{K}}, \bar{\boldsymbol{k}}$	= stiffness matrices
L	= number of lower-order modes
L_C	= number of lower-order fixed-interfacial modes
L_E	= number of lower-order free-free interfacial modes
$\boldsymbol{L_R}$	= transformation matrix in Eq. (18)
$\boldsymbol{M}, \boldsymbol{m}, \bar{\boldsymbol{M}}, \bar{\boldsymbol{m}}$	= mass matrices
$\boldsymbol{N}, \bar{\boldsymbol{N}}$	= condensation transformation matrix
$\boldsymbol{Q}, \bar{\boldsymbol{Q}}$	= general coordinate
$\boldsymbol{q}, \bar{\boldsymbol{q}}$	= modal coordinate
R	= number of rigid-body modes
$\bar{\boldsymbol{R}}(\omega^2)$	= dynamic flexibility
$\boldsymbol{T}_c$	= constraint mode matrices
$\boldsymbol{X}$	= displacement vector
Λ	= eigenvalue matrix
λ	= eigenvalue
$\Phi, \bar{\Phi}$	= modal matrix
Ψ	= residual modes
$\bar{\Psi}$	= term for residual modes
ω	= angular frequency

Subscripts

B, b	= fixed interface
$c0$	= static constraint modes
E, e	= free interface
h	= higher-order modes
i	= internal DOF
j	= boundary DOF
l	= lower-order modes
n	= substructure
R	= rigid-body DOF
α, β	= adjacent substructure

I. Introduction

SUBSTRUCTURE modal synthesis (SMS) is a kind of modeling technique. Using this technique, the degree of freedom (DOF) of a complex structure can be reduced through modal transformation. The SMS technique has been widely applied to structural dynamic analysis and has proved to be quite useful in solving large, complex structural dynamic problems, especially for structures consisting of several distinct substructures. Therefore, it has been used extensively in the aerospace and automotive industries.

It is demonstrated that substructure synthesis and all of its variants are essentially different forms of the Rayleigh–Ritz method. The variants give different eigenvalue problems depending on the trial functions used. Three classes of trial functions are of special interest, i.e., 1) eigenfunctions satisfying both the eigenvalue equation and all of the boundary conditions, 2) comparison functions satisfying all of the boundary conditions, and 3) admissible functions satisfying only the geometric boundary conditions. Meirovitch and Kwak[1] demonstrated that the ordinarily used classical Rayleigh–Ritz method has an implicit shortcoming, which can impair its convergence characteristics. If the eigenvalue problem is formulated as a variational problem, using the Rayleigh–Ritz method, a sequence of approximate solutions can be constructed in the space of admissible functions. These admissible functions can be expressed in the form of a series of trial functions, and the accuracy of the sequence of approximations is improved by increasing the number of terms in this series. The trial functions in a given series are generally taken as members of the same family of functions, but unfortunately, in many cases, solutions obtained by choosing admissible functions of the same type in this way are characterized by poor convergence. The cause can be traced to the fact that the natural boundary conditions are often very difficult to satisfy when using a relatively small number of admissible functions. The fact that the Rayleigh–Ritz theory guarantees convergence is valid provided that the admissible functions are from a complete set. This is small comfort in computational work, in which the objective is to obtain good accuracy with as few terms as possible. To overcome this predicament, Meirovitch and Kwak[1] proposed that the approximating sequence could be constructed in the space of quasicomparison functions instead of merely

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from the space of admissible functions. The quasicomparison functions are defined as linear combinations of admissible functions, which play the role of comparison functions. The general idea is to select admissible functions that can provide the possibility of satisfying the natural boundary conditions. This requires that the solution be constructed from several different types of admissible functions. It is this variety of admissible functions that permits accurate satisfaction of the natural boundary conditions with a relatively small number of terms, and this is impossible by using admissible functions of a single type. As a result, eigensolutions obtained through the use of assumed modes of quasicomparison functions exhibit superior convergence characteristics.

Clearly, a relatively small set of well-assumed modes will quickly produce satisfactory results. However, poorly assumed modes may lead to the ill conditioning of the resulting eigenequation and may even lose some important eigenvalues. In particular, it is more difficult to assume good modes for large and complex systems, so that undesirable results will more likely occur.

To solve this problem, some new modal synthesis techniques using the assumed modes directly according to the Ritz procedure were introduced, but the accuracy of system synthesis results of these techniques is not high enough. There are different assumed modes, i.e., substructure displacement representation, which is introduced by means of different mechanical analysis processes.

In this paper, using an analytical method, the substructural displacements are accurately expressed in terms of lower mixed modes, i.e., linear combinations of the lower fixed-interfacial modes and the lower free-interfacial modes. This leads to a new modal synthesis technique using mixed substructural modes. This new method is a combination of the fixed-interfacial method, free-interfacial method, and the assumed modes method of using quasicomparison functions. The key point to realize this scheme is that by analytical means the exact residual modes, which contain the effects of higher free-interface modes, are expressed accurately in terms of some lower fixed-interface modes and the accurate substructural displacements are also expressed in terms of some lower mixed modes. It can be proved that the truncation frequency f_{bN} of the substructural fixed modes and the truncation frequency f_{EN} of the substructural free-interfacial modes are two reliable criteria of truncation frequency of the system synthesis modes; these criteria indicate the accuracy and reliability of the synthesized results. Therefore, they are very important and useful in practical engineering and have great theoretical significance.

II. Background

The equation of motion of each undamped substructure may be expressed as

$$\boldsymbol{m}\ddot{\boldsymbol{X}} + \boldsymbol{k}\boldsymbol{X} = \boldsymbol{f} \tag{1}$$

where $f_i = 0$ is the force corresponding to interior DOF for the eigenvalue problem; $\boldsymbol{m}$ and $\boldsymbol{k}$ are the mass and stiffness matrices, respectively; and $\boldsymbol{X}$ and $\boldsymbol{f}$ denote the displacement vector and the force vector, respectively.

This equation can be partitioned into those associated with internal DOF i and boundary DOF j:

$$\begin{bmatrix} \boldsymbol{m}_{ii} & \boldsymbol{m}_{ij} \\ \boldsymbol{m}_{ji} & \boldsymbol{m}_{jj} \end{bmatrix} \begin{Bmatrix} \ddot{X}_i \\ \ddot{X}_j \end{Bmatrix} + \begin{bmatrix} \boldsymbol{k}_{ii} & \boldsymbol{k}_{ij} \\ \boldsymbol{k}_{ji} & \boldsymbol{k}_{jj} \end{bmatrix} \begin{Bmatrix} X_i \\ X_j \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{f}_j \end{Bmatrix} \tag{2}$$

where the subscripts i and j indicate the internal and boundary DOF, respectively.

A. Exact Displacement Vector of Substructure with Free Interfaces

According to the exact free-interface modal synthesis technique proposed by Qiu and Tan[2] and Ying et al.,[3] the exact expression for the substructure displacements $\boldsymbol{X}$ can be obtained analytically from Eq. (1) as

$$\boldsymbol{X} = (\boldsymbol{k} - \boldsymbol{m}\omega^2)^{-1}\boldsymbol{f} = \boldsymbol{\Phi}_E(\Lambda_E - \omega^2\boldsymbol{I}_E)^{-1}\boldsymbol{\Phi}_E^T\boldsymbol{f} = \boldsymbol{X}_{ER} + \boldsymbol{X}_{El} + \boldsymbol{X}_{Eh} = \boldsymbol{\Phi}_{ER}\boldsymbol{q}_{ER} + \boldsymbol{\Phi}_{El}\boldsymbol{q}_{El} + \boldsymbol{\Phi}_{Eh}\boldsymbol{q}_{Eh} = \boldsymbol{\Phi}_E\bar{\boldsymbol{q}}_E \tag{3}$$

where

$$\boldsymbol{X}_{ER} = \boldsymbol{\Phi}_{ER}\boldsymbol{q}_{ER}, \qquad \boldsymbol{X}_{El} = \boldsymbol{\Phi}_{El}\boldsymbol{q}_{El}, \qquad \boldsymbol{X}_{Eh} = \boldsymbol{\Phi}_{Eh}\boldsymbol{q}_{Eh}$$
$$\bar{\boldsymbol{q}}_E = [\boldsymbol{q}_{ER} \quad \boldsymbol{q}_{El} \quad \boldsymbol{q}_{Eh}], \qquad \boldsymbol{q}_{ER} = -\omega^{-2}\boldsymbol{I}_R\boldsymbol{\Phi}_{ER}\boldsymbol{f}$$
$$\boldsymbol{q}_{EL} = \left(\Lambda_{EL} - \omega^2\boldsymbol{I}_{El}\right)^{-1}\boldsymbol{\Phi}_{EL}^T\boldsymbol{f} \tag{4}$$
$$\boldsymbol{q}_{Eh} = \left(\Lambda_{Eh} - \omega^2\boldsymbol{I}_{Eh}\right)^{-1}\boldsymbol{\Phi}_{Eh}^T\boldsymbol{f}$$
$$\boldsymbol{\Phi}_E = [\boldsymbol{\Phi}_{ER} \quad \boldsymbol{\Phi}_{El} \quad \boldsymbol{\Phi}_{Eh}]$$

It is obvious that the complete set $\boldsymbol{\Phi}_E$ of free-interface modes is the complete set of substructure displacements. Hou[4] first proposed the classical free-interface method for modal synthesis; in this method the higher-order modes $\boldsymbol{\Phi}_{Eh}$ are completely ignored from the exact expression of the substructure displacements $\boldsymbol{X}$ in Eq. (3). The transformation from physical coordinates to modal coordinates can be performed by substituting Eq. (3) into Eq. (1) and premultiplying the obtained equation by $\boldsymbol{\Phi}_E$. Hence, the equation of motion of the substructure in modal coordinates is

$$\left\{ \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \Lambda_{EL} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \Lambda_{Eh} \end{bmatrix} - \omega^2 \begin{bmatrix} \boldsymbol{m}_{ER} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{EL} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{Eh} \end{bmatrix} \right\} \begin{Bmatrix} \boldsymbol{q}_{ER} \\ \boldsymbol{q}_{El} \\ \boldsymbol{q}_{Eh} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\Phi}_{ERj}^T\boldsymbol{f}_j \\ \boldsymbol{\Phi}_{Elj}^T\boldsymbol{f}_j \\ \boldsymbol{\Phi}_{Ehj}^T\boldsymbol{f}_j \end{Bmatrix} \tag{5}$$

where $\boldsymbol{m}_{ER} = \boldsymbol{\Phi}_{ER}^T\boldsymbol{m}\boldsymbol{\Phi}_{ER}$.

Combining Eq. (4) with the third-row equation in Eq. (5) gives the exact higher-mode response $\boldsymbol{X}_{Eh}$ as

$$\boldsymbol{X}_{Eh} = \boldsymbol{\Phi}_{Eh}\left(\Lambda_{Eh} - \omega^2\boldsymbol{I}_{Eh}\right)\boldsymbol{\Phi}_{Ehj}^T\boldsymbol{f}_j = \boldsymbol{\Psi}_j\boldsymbol{f}_j \tag{6}$$

and the substructure displacement $\boldsymbol{X}$ is obtained as

$$\boldsymbol{X} = \boldsymbol{\Phi}_{ER}\boldsymbol{q}_{ER} + \boldsymbol{\Phi}_{El}\boldsymbol{q}_{El} + \boldsymbol{\Psi}_j\boldsymbol{f}_j = \boldsymbol{\Phi}_e\bar{\boldsymbol{q}}_e \tag{7}$$

where

$$\boldsymbol{\Phi}_e = [\boldsymbol{\Phi}_{ER} \quad \boldsymbol{\Phi}_{EL} \quad \boldsymbol{\Psi}_j], \qquad \bar{\boldsymbol{q}}_e = [\boldsymbol{q}_{ER} \quad \boldsymbol{q}_{EL} \quad \boldsymbol{f}_j] \tag{8}$$

It also leads to a nonlinear synthesis equation. The exact residual mode $\boldsymbol{\Psi}_j$ contains the effects of higher modes, and it is expressed in three parts as follows:

$$\boldsymbol{\Psi}_j = \boldsymbol{\Phi}_{Eh}\left(\Lambda_{Eh} - \omega^2\boldsymbol{I}_{Eh}\right)\boldsymbol{\Phi}_{Ehj}^T = \boldsymbol{\Psi}_{j1} + \omega^2\boldsymbol{\Psi}_{j12} + \omega^4\boldsymbol{\Psi}_{j3} = \boldsymbol{\Psi}_{j2} + \omega^4\boldsymbol{\Psi}_{j3} \tag{9}$$

where

$$\boldsymbol{\Psi}_{j2} = \boldsymbol{\Psi}_{j1} + \omega^2\boldsymbol{\Psi}_{j12}, \qquad \boldsymbol{\Psi}_{j1} = \boldsymbol{\Phi}_{Eh}\Lambda_{Eh}^{-1}\boldsymbol{\Phi}_{Ehj}^T$$
$$\boldsymbol{\Psi}_{j12} = \boldsymbol{\Psi}_{h1}\boldsymbol{m}\boldsymbol{\Psi}_{j1}, \qquad \boldsymbol{\Psi}_{j3} = \boldsymbol{\Psi}_h\boldsymbol{m}\boldsymbol{\Psi}_{j12} \tag{10}$$

The first-order approximation $\boldsymbol{\Psi}_{j1}$ is MacNeal's approximation.[5] The second-order approximation $\boldsymbol{\Psi}_{j2}$ is Rubin's approximation,[6] as used by Craig and Chang[7] and Wang et al.[8] The third part $\boldsymbol{\Psi}_{j3}$ is the exact residual term of Rubin's approximation.

Thus, Refs. 2–8 form a systematic logical progress of the substructure modal synthesis method.

B. Exact Displacement Vector of Fixed Interface Substructure

According to the exact fixed interface modal synthesis technique proposed by Qiu et al.,[9] from the upper half of Eq. (2),

$$\boldsymbol{X}_i = -\left(\boldsymbol{k}_{ii} - \omega^2\boldsymbol{m}_{ii}\right)^{-1}\left(\boldsymbol{k}_{ij} - \omega^2\boldsymbol{m}_{ij}\right)\boldsymbol{X}_j$$
$$= -\boldsymbol{k}_{ii}^{-1}\boldsymbol{k}_{ij}\boldsymbol{X}_i + \omega^2\boldsymbol{\Phi}_{bi}\left(\Lambda_b - \omega^2\boldsymbol{I}_b\right)^{-1}\boldsymbol{\Phi}_b^T\boldsymbol{m}\boldsymbol{\Phi}_{c0}\boldsymbol{X}_i$$

where

$$\Phi_{c0} = \begin{bmatrix} t_{c0} \\ I \end{bmatrix}, \qquad t_{c0} = -k_{ii}^{-1} k_{ij} \tag{11}$$

The exact expression for the substructure displacements X can be obtained analytically as

$$X = \begin{bmatrix} X_i \\ X_j \end{bmatrix} = X_{bh} + X_{bl} + X_{c0} = \Phi_c \bar{q}_c = \begin{bmatrix} X_{bi} \\ 0 \end{bmatrix} + \begin{bmatrix} t_{c0} \\ I \end{bmatrix} X_j \tag{12}$$

where

$$X_{bh} = \Phi_{bh} q_{bh}, \qquad X_{bl} = \Phi_{bl} q_{bl}, \qquad X_{c0} = \Phi_{c0} X_j$$
$$X_{bi} = \Phi_{bli} q_{bl} + \Phi_{bhi} q_{bh}, \qquad \Phi_c = [\Phi_{bh} \quad \Phi_{bl} \quad \Phi_{c0}]$$
$$\bar{q}_c = \left[q_{bh}^T \quad q_{bl}^T \quad X_j^T\right]^T, \qquad q_{bh} = \omega^2 \left(\Lambda_b - \omega^2 I_b\right) \Phi_{bh}^T m \Phi_{c0} X_j$$
$$q_{bl} = \omega^2 \left(\Lambda_b - \omega^2 I_b\right) \Phi_{bl}^T m \Phi_{c0} X_j \tag{13}$$

Equation (12) means that the complete set of substructure displacements X is equivalent to the complete set Φ_b of fixed-interface normal modes plus the static constraint modes Φ_{c0}. When the higher-order modes Φ_{bh} are neglected completely, the exact substructure displacement in Eq. (12) becomes the substructure displacement $\bar{X}$ obtained using Craig and Bampton's fixed interface method[10] as

$$\bar{X} = \Phi_{bl} q_{bl} + \Phi_{c0} X_j \tag{14}$$

III. Mixed Substructural Modes

A. Relationship Between Rigid-Body Modes and Static Constraint Modes

The interface constrained force f_{c0j} corresponding to the static constraint modes Φ_{c0} can be expressed as

$$f_{c0j} = k_{jj} - k_{ji} k_{ii}^{-1} k_{ij} \tag{15}$$

so that

$$k\Phi_{c0} = \begin{bmatrix} 0 \\ f_{c0j} \end{bmatrix} \tag{16}$$

For any substructure with fixed interfaces, the number of interface DOF j cannot be less than the number of rigid-body DOF R of the substructure with its interfaces free. There are statically determinate constrained modes Φ_{cR} and statically indeterminate (redundant) constrained modes Φ_{cc}. Let

$$\Phi_{c0} = [\Phi_{cR} \quad \Phi_{cc}], \qquad f_{c0j} = [f_{cRj} \quad f_{ccj}] \tag{17}$$

The interface force f_{cRj} of the static determinate constrained modes must satisfy the following static equilibrium equations of order R:

$$f_{cRj} L_R = 0 \tag{18}$$

There must exist an $R \times j$ matrix L_R and a $j \times j$ matrix L_J

$$L_J = \begin{bmatrix} L_R \\ 0 \end{bmatrix} \tag{19}$$

The f_{c0j} on the right-hand side of Eq. (16) must satisfy the equilibrium equations, which are also of order R:

$$\begin{bmatrix} 0 \\ f_{c0j} \end{bmatrix} L_J = \begin{bmatrix} 0 & 0 \\ f_{cRj} & f_{ccj} \end{bmatrix} L_J = \begin{bmatrix} 0 \\ f_{cRj} L_R \end{bmatrix} = 0 \tag{20}$$

Substituting Eq. (16) into Eq. (20) gives

$$\begin{bmatrix} 0 \\ f_{c0j} \end{bmatrix} L_J = k\Phi_{c0} L_J = k\Phi_{ER} = 0 \tag{21}$$

where

$$\Phi_{ER} = \Phi_{c0} L_J = [\Phi_{cR} \quad \Phi_{cc}] \begin{bmatrix} L_R \\ 0 \end{bmatrix} = \Phi_{cR} L_R \tag{22}$$

Equation (21) shows that by definition Φ_{ER} must be rigid-body modes. Hence, it can be seen from Eq. (22) that the rigid-body modes Φ_{ER} can be represented as a linear combination of the static determinate constrained modes. From Eq. (22),

$$\Phi_{cR} = \Phi_{ER} \bar{q}_{cR}, \qquad \bar{q}_{cR} = L_R^T \left(L_R L_R^T\right)^{-1} \tag{23}$$

Substituting Eqs. (23) and (17) into Eq. (14) gives another form of the substructure displacement $\bar{X}$ as

$$\bar{X} = \Phi_{bl} q_{bl} + \Phi_{ER} q_{ER} + \Phi_{cc} X_j, \qquad q_{ER} = \bar{q}_{cR} X_j \tag{24}$$

Equation (24) was first proposed by Hurty[11]; it contains three types of substructure modes, namely, lower normal modes Φ_{bl} for a fixed-interface, rigid-body modes Φ_{ER}, and redundant constrained modes Φ_{cc}. This method was later modified by Craig and Bampton,[10] who used only the constrained modes and the lower-order normal modes with fixed interface in their formulation (14). Equation (24) indicates that Craig–Bampton's substructure displacements (14) is completely equivalent to Hurty's substructure displacement (24). Actually, Craig–Bampton's method and Hurty's method are the same, and they are herein called the CBH method. For a boundary with many redundant DOF, it is not entirely clear which DOF should be treated as statically determinate and which as redundant. Craig–Bampton's method does not require the engineer to make such distinctions, and this is the advantage of their method.

If the rigid-body DOF R does not equal zero, the interfacial DOF j equals the rigid-body DOF R, and the redundant DOF equals zero, then Eq. (24) can be rewritten as

$$\bar{X} = \Phi_{bl} q_{bl} + \Phi_{ER} q_{ER} \tag{25}$$

B. Expressions of Fixed-Interfacial Modes in Terms of Lower Free-Interfacial Modes and Exact Residual Modes

Substituting Eq. (22) into Eq. (3), by using the analytical method, the substructural fixed-interfacial modes can be represented exactly by means of the free-interfacial modes. In fact, only lower free-interfacial modes can be obtained from modal analysis or modal test. Therefore, it is desirable that the exact residual modes be represented by means of lower fixed-interfacial modes. For this purpose, substituting Eq. (22) into Eq. (7), the following equation with a system synthesis frequency ω^* can be obtained:

$$X = \begin{bmatrix} t_{c0} \\ I \end{bmatrix} X_j + \begin{bmatrix} \Phi_{Elc} \\ 0 \end{bmatrix} q_{EL} + \begin{bmatrix} \Psi_{jc}(\omega^{*2}) \\ 0 \end{bmatrix} f_j \tag{26}$$

where

$$\Phi_{Elc} = \Phi_{Eli} - t_{c0} \Phi_{Elj}, \qquad \Psi_{jc}(\omega^{*2}) = \Psi_{ji}(\omega^{*2}) - t_{c0} \Psi_{jj}(\omega^{*2})$$
$$X_j = L_J q_{ER} + \Phi_{Elj} q_{El} + \Psi_{jj}(\omega^{*2}) f_i \tag{27}$$

From Eqs. (12) and (26), the interior displacement of the fixed-interfacial substructure can be expressed in terms of Φ_{Elc}, which are the lower modes of the free-interface substructure, and $\Psi_{jc}(\omega^{*2})$, which are the exact residual modes with a system synthesis frequency ω^*:

$$X_{bi} = \Phi_{Elc} q_{EL} + \Psi_{jc}(\omega^{*2}) f_j = \Phi_B(\omega^{*2}) q_B \tag{28}$$

$$\Phi_B(\omega^{*2}) = \left[\Phi_{Elc} \quad \Psi_{jc}(\omega^{*2})\right], \qquad q_B = \left[q_{El}^T \quad f_j^T\right]^T \tag{29}$$

The eigenvalue equation associated with Eq. (28) is

$$\left(k_B - \omega^2 m_B\right) q_B = 0 \tag{30}$$

where

$$k_B = \Phi_B^T(\omega^{*2}) k_{ii} \Phi_B(\omega^{*2}), \qquad m_B = \Phi_B^T(\omega^{*2}) m_{ii} \Phi_B(\omega^{*2}) \tag{30a}$$

Substituting the eigenvalue matrix Λ_B and eigenvector matrix q_{Bc} of Eq. (30) into Eq. (28), the kth mode of Eq. (30) can be obtained as

$$\Phi_{bik} = \Phi_{Elc} q_{Eck} + \Psi_{jc}(\omega^{*2}) f_{jck}, \qquad k = 1, 2, \ldots, L_c, \ldots, N \tag{31}$$

Then, the lower fixed-interfacial modes can be expressed as

$$\boldsymbol{\Phi}_{bli} = \boldsymbol{\Phi}_{Elc}\boldsymbol{q}_{Ecl} + \boldsymbol{\Psi}_{jc}(\omega^{*2})\boldsymbol{f}_{jcl} \tag{32}$$

where

$$\boldsymbol{f}_{jcl} = [\boldsymbol{f}_{jc1} \quad \boldsymbol{f}_{jc2} \quad \cdots \quad \boldsymbol{f}_{jcL}] \tag{33}$$

C. Expressions of Substructural Displacement in Terms of Lower Mixed Modes

For any substructure, the lower modes are usually obtained more easily than the higher modes, either from tests or from numerical analysis, so that it is easier to find the lower fixed-interfacial modes $\boldsymbol{\Phi}_{bl}$ than to find the higher free-interfacial modes $\boldsymbol{\Phi}_{Eh}$. Then, Eq. (32) can be rewritten as

$$\boldsymbol{\Psi}_{jc}(\omega^{*2})\boldsymbol{f}_{jcl} = \boldsymbol{\Phi}_{bli} - \boldsymbol{\Phi}_{Elc}\boldsymbol{q}_{Ecl} \tag{34}$$

Assuming that the number $\boldsymbol{L}_c$ of the lower fixed-interfacial modes $\boldsymbol{\Phi}_{bli}$ is equal to or larger than the interfacial DOF j and $(\boldsymbol{f}_{jcl}\boldsymbol{f}_{jcl}^T)^{-1}$ exists, the least square approximation $\bar{\boldsymbol{\Psi}}_{jc}$ can be obtained from Eq. (34) as

$$\bar{\boldsymbol{\Psi}}_{jc} = (\boldsymbol{\Phi}_{bli} - \boldsymbol{\Phi}_{Elc}\boldsymbol{q}_{Ecl})\boldsymbol{f}_{jcl}^T\left(\boldsymbol{f}_{jcl}\boldsymbol{f}_{jcl}^T\right)^{-1} \tag{35}$$

The approximation $\bar{\boldsymbol{\Psi}}_{jc}$ may be used instead of $\boldsymbol{\Psi}_{jc}(\omega^{*2})$, which are the exact residual modes. Substituting Eq. (35) into Eq. (26) gives

$$\begin{aligned}\boldsymbol{X} &= \begin{bmatrix}\boldsymbol{t}_{c0}\\ \boldsymbol{I}\end{bmatrix}\boldsymbol{X}_j + \begin{bmatrix}\boldsymbol{\Phi}_{Elc}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{q}_{El} + \begin{bmatrix}\boldsymbol{\Phi}_{bli}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{q}_b - \begin{bmatrix}\boldsymbol{\Phi}_{Elc}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{q}_{Ecl}\boldsymbol{q}_b\\ &= \boldsymbol{\Phi}_{c0}\boldsymbol{q}_0 + \boldsymbol{\Phi}_{bl}\boldsymbol{q}_b + \boldsymbol{\Phi}_{El}\boldsymbol{q}_E\end{aligned} \tag{36}$$

where

$$\boldsymbol{q}_0 = \boldsymbol{X}_j - \boldsymbol{\Phi}_{Elj}\boldsymbol{q}_E, \qquad \boldsymbol{q}_b = \boldsymbol{f}_{jc}^T\left(\boldsymbol{f}_{jc}\boldsymbol{f}_{jc}^T\right)^{-1}\boldsymbol{f}_j$$
$$\boldsymbol{q}_E = \boldsymbol{q}_{EL} - \boldsymbol{q}_{Ecl}\boldsymbol{q}_b \tag{37}$$

The substructural displacement $\boldsymbol{X}$ is accurately expressed in terms of some lower mixed modes, i.e., the linear combinations of the static constraint modes $\boldsymbol{\Phi}_{c0}$, the lower fixed-interface modes $\boldsymbol{\Phi}_{bl}$, and the lower free-interface modes $\boldsymbol{\Phi}_{El}$.

In Eq. (36) there are two finite series of modes, so that there are two important parameters: one is the truncation frequency f_{EN} ($f_{EN} = \lambda_{EN}/2\pi$) of lower free-interfacial modes $\boldsymbol{\Phi}_{El}$, and the other is the truncation frequency f_{bN} ($f_{bN} = \lambda_{bN}/2\pi$) of lower fixed-interfacial modes $\boldsymbol{\Phi}_{bl}$.

It is obvious that the truncation frequency f_{EN} is a reliable criterion for the accuracy of expressions of substructural displacement, i.e., when frequency f ($f = \omega/2\pi$) is lower than f_{EN}, the substructural displacement $\boldsymbol{X}$ can be represented accurately by Eq. (36), and the accuracy of expressions of substructural displacement is higher than the one for frequency f higher than f_{EN}.

The key point of using Eq. (36) to obtain substructural displacement is that by analytical means the term $\boldsymbol{\Psi}_{jc}(\omega^{*2})$ of exact residual modes, which contains the effects of the higher free-interface modes, can be expressed in terms of some lower fixed-interface modes by means of the accurate expressions in Eq. (35). It follows that there is another reliable criterion for the accuracy of expressions of substructural displacement, i.e., when $L_c \leq j$, the term $\boldsymbol{\Psi}_{jc}(\omega^{*2})$ of the exact residual modes can be represented accurately by Eq. (35) using the lower fixed-interfacial modes $\boldsymbol{\Phi}_{bl}$, and the substructural displacement $\boldsymbol{X}$ can be represented accurately by Eq. (36).

Substituting Eqs. (17) and (23) into Eq. (36), the substructural displacement can be represented accurately by means of mixed substructural modes, i.e., the linear combinations of the lower free-interfacial modes $\boldsymbol{\Phi}_{ER}$ and $\boldsymbol{\Phi}_{El}$, the redundant constrained modes $\boldsymbol{\Phi}_{cc}$, and the lower fixed-interfacial modes $\boldsymbol{\Phi}_{bl}$, that is,

$$\boldsymbol{X} = \boldsymbol{\Phi}_{ER}\boldsymbol{q}_R + \boldsymbol{\Phi}_{cc}\boldsymbol{q}_{0c} + \boldsymbol{\Phi}_{bl}\boldsymbol{q}_c + \boldsymbol{\Phi}_{El}\boldsymbol{q}_E \tag{38}$$

$$\boldsymbol{q}_R = \bar{\boldsymbol{q}}_{cR}\boldsymbol{q}_{0R}, \qquad \boldsymbol{q}_0 = \begin{Bmatrix}\boldsymbol{q}_{0R}\\ \boldsymbol{q}_{0c}\end{Bmatrix} \tag{39}$$

When the lower free-interface modes $\boldsymbol{\Phi}_{El}$ completely vanish, Eq. (36) becomes Craig and Bampton's[10] substructure displacement (14) and Eq. (38) becomes Hurty's[11] substructural displacement (24). Hence, Craig–Bampton's method and Hurty's method are equivalent to the proposed method for some special cases. When the redundant constrained modes $\boldsymbol{\Phi}_{cc}$ and the lower fixed-interfacial modes $\boldsymbol{\Phi}_{bl}$ completely vanish, Eq. (38) becomes Hou's[4] expression for substructural displacement. Hence, Hou's method is another special case of the proposed method.

If the rigid-body DOF R is not equal to zero, the interfacial DOF j equals the rigid-body DOF R; meanwhile the redundant constrained modes $\boldsymbol{\Phi}_{cc}$ vanish, and the substructural displacement can be represented accurately by means of mixed substructural modes, i.e., the linear combinations of the lower free-interfacial modes $\boldsymbol{\Phi}_{ER}$ and $\boldsymbol{\Phi}_{El}$ and the lower fixed-interfacial modes $\boldsymbol{\Phi}_{bl}$, that is,

$$\boldsymbol{X} = \boldsymbol{\Phi}_{ER}\boldsymbol{q}_R + \boldsymbol{\Phi}_{bl}\boldsymbol{q}_b + \boldsymbol{\Phi}_{El}\boldsymbol{q}_E \tag{40}$$

IV. Mixed Modal Synthesis Method

A. Substructure Modal Coordinate Transformation

Based on Eq. (36), the displacement of the substructure can be represented approximately using mixed modes, i.e., the linear combination of the lower free-interfacial modes $\boldsymbol{\Phi}_{El}$, static constrained modes $\boldsymbol{\Phi}_{c0}$, and fixed-interfacial modes $\boldsymbol{\Phi}_{bl}$, as

$$\boldsymbol{X} = \boldsymbol{\Phi}\boldsymbol{q} \tag{41}$$

where

$$\boldsymbol{\Phi} = [\boldsymbol{\Phi}_{c0} \quad \boldsymbol{\Phi}_{El} \quad \boldsymbol{\Phi}_{bl}], \qquad \boldsymbol{q} = \left[\boldsymbol{q}_0^T \quad \boldsymbol{q}_E^T \quad \boldsymbol{q}_b^T\right]^T \tag{42}$$

Taking Eq. (41) as the Ritz assumption of displacement $\boldsymbol{X}$, Eq. (1) becomes the substructure dynamic equation

$$\left(\boldsymbol{k}_n - \omega^2\boldsymbol{m}_n\right)\boldsymbol{q} = \boldsymbol{f}_n \tag{43}$$

where

$$\boldsymbol{k}_n = \begin{bmatrix}\boldsymbol{k}_0 & \boldsymbol{k}_{0E} & \boldsymbol{0}\\ \boldsymbol{k}_{E0} & \Lambda_{El} & \boldsymbol{k}_{Eb}\\ \boldsymbol{0} & \boldsymbol{k}_{bE} & \Lambda_{bl}\end{bmatrix}, \qquad \boldsymbol{m}_n = \begin{bmatrix}\boldsymbol{m}_0 & \boldsymbol{m}_{0E} & \boldsymbol{m}_{0b}\\ \boldsymbol{m}_{E0} & \boldsymbol{I}_{El} & \boldsymbol{m}_{Eb}\\ \boldsymbol{m}_{b0} & \boldsymbol{m}_{bE} & \boldsymbol{I}_{bl}\end{bmatrix}$$
$$\boldsymbol{f}_n = \begin{Bmatrix}\boldsymbol{\Phi}_{ERj}^T\boldsymbol{f}_j\\ \boldsymbol{\Phi}_{Elj}^T\boldsymbol{f}_j\\ \boldsymbol{0}\end{Bmatrix} \tag{44}$$

$$\begin{gathered}\boldsymbol{m}_0 = \boldsymbol{\Phi}_{c0}^T\boldsymbol{m}\boldsymbol{\Phi}_{c0}, \qquad \boldsymbol{I}_{bl} = \boldsymbol{\Phi}_{bl}^T\boldsymbol{m}\boldsymbol{\Phi}_{bl}, \qquad \boldsymbol{I}_{El} = \boldsymbol{\Phi}_{El}^T\boldsymbol{m}\boldsymbol{\Phi}_{El}\\ \boldsymbol{k}_0 = \boldsymbol{\Phi}_{c0}^T\boldsymbol{k}\boldsymbol{\Phi}_{c0}, \qquad \Lambda_{bl} = \boldsymbol{\Phi}_{bl}^T\boldsymbol{k}\boldsymbol{\Phi}_{bl}, \qquad \Lambda_{El} = \boldsymbol{\Phi}_{El}^T\boldsymbol{k}\boldsymbol{\Phi}_{El}\\ \boldsymbol{k}_{0b} = \boldsymbol{k}_{b0}^T = \boldsymbol{\Phi}_{c0}^T\boldsymbol{k}\boldsymbol{\Phi}_{bl} = \boldsymbol{0}, \qquad \boldsymbol{m}_{0b} = \boldsymbol{m}_{b0}^T = \boldsymbol{\Phi}_{c0}^T\boldsymbol{m}\boldsymbol{\Phi}_{bl}\\ \boldsymbol{k}_{0E} = \boldsymbol{k}_{E0}^T = \boldsymbol{\Phi}_{c0}^T\boldsymbol{k}\boldsymbol{\Phi}_{El}, \qquad \boldsymbol{m}_{0E} = \boldsymbol{m}_{E0}^T = \boldsymbol{\Phi}_{c0}^T\boldsymbol{m}\boldsymbol{\Phi}_{El}\\ \boldsymbol{m}_{Eb} = \boldsymbol{k}_{bE}^T = \boldsymbol{\Phi}_{El}^T\boldsymbol{m}\boldsymbol{\Phi}_{bl}, \qquad \boldsymbol{k}_{Eb} = \boldsymbol{k}_{bE}^T = \boldsymbol{\Phi}_{El}^T\boldsymbol{k}\boldsymbol{\Phi}_{bl}\end{gathered} \tag{45}$$

B. Mixed Modal Synthesis Technique

For simplicity, only adjacent substructures α and β are taken to describe the mixed modal synthesis technique. It is not difficult to extend the procedure to the case with more than two substructures. Using Eq. (43) for both substructures and putting them together yields

$$\left(\begin{bmatrix}\boldsymbol{k}_{n\alpha} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{k}_{n\beta}\end{bmatrix} - \omega^2\begin{bmatrix}\boldsymbol{m}_{n\alpha} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{m}_{n\beta}\end{bmatrix}\right)\begin{Bmatrix}\boldsymbol{q}_\alpha\\ \boldsymbol{q}_\beta\end{Bmatrix} = \begin{Bmatrix}\boldsymbol{f}_{n\alpha}\\ \boldsymbol{f}_{n\beta}\end{Bmatrix} \tag{46}$$

Based on the compatibility relations, the interface displacement $\boldsymbol{X}_{j\alpha} = \boldsymbol{X}_{j\beta}$, one obtains

$$\boldsymbol{q}_{0\alpha} = \boldsymbol{N}_1\boldsymbol{Q} \tag{47}$$

Then, execute the condensation transformation

$$\boldsymbol{q}_{\alpha\beta} = \boldsymbol{N}\boldsymbol{Q} \tag{48}$$

where

$$\boldsymbol{q}_{\alpha\beta} = \begin{bmatrix}\boldsymbol{q}_\alpha^T & \boldsymbol{q}_\beta^T\end{bmatrix}^T = \begin{bmatrix}\boldsymbol{q}_{0\alpha}^T & \boldsymbol{q}_{E\alpha}^T & \boldsymbol{q}_{b\alpha}^T & \boldsymbol{q}_{0\beta}^T & \boldsymbol{q}_{E\beta}^T & \boldsymbol{q}_{b\beta}^T\end{bmatrix}^T$$

$$\boldsymbol{Q} = \begin{bmatrix}\boldsymbol{q}_{E\alpha}^T & \boldsymbol{q}_{b\alpha}^T & \boldsymbol{q}_{0\beta}^T & \boldsymbol{q}_{E\beta}^T & \boldsymbol{q}_{b\beta}^T\end{bmatrix}^T \tag{49}$$

$$\boldsymbol{N} = \begin{bmatrix}\boldsymbol{N}_1 \\ \boldsymbol{I}\end{bmatrix}, \qquad \boldsymbol{N}_1 = [-\boldsymbol{\Phi}_{Elj\alpha} \quad \boldsymbol{0} \quad \boldsymbol{I}_{j\beta} \quad \boldsymbol{\Phi}_{Elj\beta} \quad \boldsymbol{0}] \tag{50}$$

After using the condensation transformation Eq. (48), Eq. (46) becomes the modal synthesis equation

$$(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{Q} = \boldsymbol{0} \tag{51}$$

where

$$\boldsymbol{K} = \boldsymbol{N}^T \begin{bmatrix}\boldsymbol{k}_{n\alpha} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{k}_{n\beta}\end{bmatrix}\boldsymbol{N}, \qquad \boldsymbol{M} = \boldsymbol{N}^T \begin{bmatrix}\boldsymbol{m}_{n\alpha} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{m}_{n\beta}\end{bmatrix}\boldsymbol{N} \tag{52}$$

It is obvious that Eq. (51) is a linear eigenvalue equation, which can be solved easily.

If the number of interface DOF j equals the number of rigid-body DOF R, based on Eq. (39), using the mixed modes $\boldsymbol{\Phi}_{El}$, $\boldsymbol{\Phi}_{ER}$, and $\boldsymbol{\Phi}_{bl}$, the substructural displacement can be expressed as

$$\boldsymbol{X} = \bar{\boldsymbol{\Phi}}\bar{\boldsymbol{q}} \tag{53}$$

where

$$\bar{\boldsymbol{\Phi}} = [\boldsymbol{\Phi}_{ER} \quad \boldsymbol{\Phi}_{El} \quad \boldsymbol{\Phi}_{bl}], \qquad \bar{\boldsymbol{q}} = \begin{bmatrix}\boldsymbol{q}_R^T & \boldsymbol{q}_E^T & \boldsymbol{q}_b^T\end{bmatrix}^T \tag{54}$$

Taking the displacement Eq. (53) as the Ritz assumption instead of Eq. (41) and proceeding analogously to the derivations of Eqs. (42–50) gives Eqs. (55–57) as follows.

The substructure dynamic equation is

$$\left(\bar{\boldsymbol{k}}_n - \omega^2\bar{\boldsymbol{m}}_n\right)\bar{\boldsymbol{q}} = \bar{\boldsymbol{f}}_n \tag{55}$$

and the modal synthesis equation for the system structure is

$$\left(\bar{\boldsymbol{K}} - \omega^2\bar{\boldsymbol{M}}\right)\bar{\boldsymbol{Q}} = \boldsymbol{0}, \qquad \bar{\boldsymbol{Q}} = \begin{bmatrix}\boldsymbol{q}_{E\alpha}^T & \boldsymbol{q}_{b\alpha}^T & \boldsymbol{q}_{R\beta}^T & \boldsymbol{q}_{E\beta}^T & \boldsymbol{q}_{b\beta}^T\end{bmatrix}^T \tag{56}$$

Using Eqs. (44) and (45) and $\boldsymbol{k}\boldsymbol{\Phi}_{ER} = \boldsymbol{0}$, one can obtain

$$\bar{\boldsymbol{k}}_n = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_{Eb}\end{bmatrix}, \qquad \bar{\boldsymbol{m}}_n = \begin{bmatrix}\boldsymbol{I}_R & \boldsymbol{0} & \bar{\boldsymbol{m}}_{Rb} \\ \boldsymbol{0} & \boldsymbol{I}_{El} & \bar{\boldsymbol{m}}_{Eb} \\ \bar{\boldsymbol{m}}_{bR} & \bar{\boldsymbol{m}}_{bE} & \boldsymbol{I}_{bl}\end{bmatrix}$$

$$\bar{\boldsymbol{f}}_n = \begin{bmatrix}\boldsymbol{\Phi}_{ERj}^T\boldsymbol{f}_j \\ \boldsymbol{\Phi}_{Elj}^T\boldsymbol{f}_j \\ \boldsymbol{0}\end{bmatrix}, \qquad \boldsymbol{K}_{Eb} = \begin{bmatrix}\boldsymbol{\Lambda}_{El} & \bar{\boldsymbol{k}}_{Eb} \\ \bar{\boldsymbol{k}}_{bE} & \boldsymbol{\Lambda}_{bl}\end{bmatrix}$$

$$\bar{\boldsymbol{k}}_{Eb} = \bar{\boldsymbol{k}}_{bE}^T = \boldsymbol{\Phi}_{El}^T\boldsymbol{k}\boldsymbol{\Phi}_{bl}, \qquad \bar{\boldsymbol{m}}_{RE} = \bar{\boldsymbol{m}}_{ER}^T = \boldsymbol{\Phi}_{ER}^T\boldsymbol{m}\boldsymbol{\Phi}_{El} = \boldsymbol{0}$$

$$\bar{\boldsymbol{m}}_{Rb} = \bar{\boldsymbol{m}}_{bR}^T = \boldsymbol{\Phi}_{ER}^T\boldsymbol{m}\boldsymbol{\Phi}_{bl}, \qquad \bar{\boldsymbol{m}}_{Eb} = \bar{\boldsymbol{m}}_{bE}^T = \boldsymbol{\Phi}_{El}^T\boldsymbol{m}\boldsymbol{\Phi}_{bl} \tag{57}$$

$$\bar{\boldsymbol{K}} = \bar{\boldsymbol{N}}^T \begin{bmatrix}\bar{\boldsymbol{k}}_{n\alpha} & \boldsymbol{0} \\ \boldsymbol{0} & \bar{\boldsymbol{k}}_{n\beta}\end{bmatrix}\bar{\boldsymbol{N}} = \begin{bmatrix}\boldsymbol{K}_{Eb\alpha} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{K}_{Eb\beta}\end{bmatrix}$$

$$\bar{\boldsymbol{M}} = \bar{\boldsymbol{N}}^T \begin{bmatrix}\bar{\boldsymbol{m}}_{n\alpha} & \boldsymbol{0} \\ \boldsymbol{0} & \bar{\boldsymbol{m}}_{n\beta}\end{bmatrix}\bar{\boldsymbol{N}}, \qquad \bar{\boldsymbol{q}} = \bar{\boldsymbol{Q}}\bar{\boldsymbol{N}}; \qquad \bar{\boldsymbol{N}} = \begin{bmatrix}\bar{\boldsymbol{N}}_1 \\ \boldsymbol{I}\end{bmatrix}$$

$$\bar{\boldsymbol{N}}_1 = \boldsymbol{\Phi}_{ERj\alpha}^{-1}[-\boldsymbol{\Phi}_{Elj\alpha} \quad \boldsymbol{0} \quad \boldsymbol{\Phi}_{ERj\beta} \quad \boldsymbol{\Phi}_{Elj\beta} \quad \boldsymbol{0}]$$

Let the truncation frequencies f_{EN} and f_{bN} for every substructure be the same and $f_{EN} \geq f_{bN}$; there are two reliable criteria for truncation frequency of the system synthesis modes: one is the truncation frequency f_{EN} for lower free-interfacial modes $\boldsymbol{\Phi}_{El}$, and the other is the truncation frequency f_{bN} for lower fixed-interfacial modes $\boldsymbol{\Phi}_{bl}$.

When $L_c \geq j$, and frequency f is less than the truncation frequencies f_{bN} and f_{EN}, the term of residual modes can be represented accurately by Eq. (35), and the substructural displacement $\boldsymbol{X}$ can be represented accurately by Eq. (36). Therefore, the accuracy for system synthesis frequencies f, which is less than the truncation frequency f_{bN}, is very high. When the synthesis frequency f is higher than the truncation frequency f_{bN} and less than the truncation frequency f_{EN}, the accuracy of synthesis frequency f is also good.

These two criteria indicate the accuracy and reliability of system synthesis results, which are very important and useful in practical engineering. Now, according to the requirement of accuracy on system synthesis results of interest, the truncation frequency of the system synthesis modes could be determined, and then the truncation frequency f_{EN} of free-interfacial modes $\boldsymbol{\Phi}_{El}$ and the truncation frequency f_{bN} of fixed-interfacial modes $\boldsymbol{\Phi}_{bl}$ for every substructure could also be determined. The two truncation frequencies f_{bN} and f_{EN} of substructural modes provide accurate expressions of substructural displacement $\boldsymbol{X}$ with a relatively small number of mixed modes, and only linear synthesis equation (56) is involved in this proposed method. Hence, by using this proposed method the synthesis procedure is very easy and the calculation is very simple, so that the present method not only has simple form and high precision but also possesses high efficiency and reliability.

V. Numerical Example

A. Example 1

A launch vehicle is simplified into a free-free beam and a fixed-free beam (Fig. 1) for the vehicle material Young's modulus $E = 2 \times 10^9$ Pa, the mass density $\rho = 4 \times 10^3$ kg/m^3, the sectional moment of inertia $I = 2 \times 10^{-8}$ m^4, and the cross-sectional area $A = 1 \times 10^{-3}$ m^2.

To illustrate the efficiency of the proposed method without going into computational complications, this vehicle is modeled as a uniform spatial cylindrical beam. There is the problem of repeated eigenmodes in the spatial cylindrical beam; in Tables 1–4, each frequency represents a pair of repeated frequencies. The calculated results based on the finite element method (FEM) without any reduction are used as the reference standard solutions for illustrating the computational accuracy of the proposed method.

The whole free-free beam with length $L = 1.0$ m was divided into two substructures with equal length $l = 0.5$ m (Fig. 1). Each

Table 1 Synthesis frequencies of free-free beam using proposed method, where ω (1/s) = $2\pi f$

	CBH		Craig		Mixed		Standard
i	f, Hz	Error, %	f, Hz	Error, %	f, Hz	Error, %	f, Hz
1, 2	0		0		0		0
3	11.2321	0.0000	11.2322	0.0009	11.2321	0.0000	11.2321
4	30.8708	0.0000	30.8706	0.0006	30.8707	0.0002	30.8708
5	60.3041	0.0237	60.2957	0.0098	60.2897	0.0002	60.2898
6[a]	99.2070	0.0102	99.2104	0.0136	99.1978	0.0009	99.1969
7	147.588	0.1475	148.380	0.6847	147.394	0.0153	147.371
8	204.754	0.0991	218.934	7.0315	206.011	0.7137	204.551
9	272.038	0.5861			286.194	5.8203	270.453

[a]Truncation frequency of synthesis modes (maximum error for the mixed method is less than 0.001%).

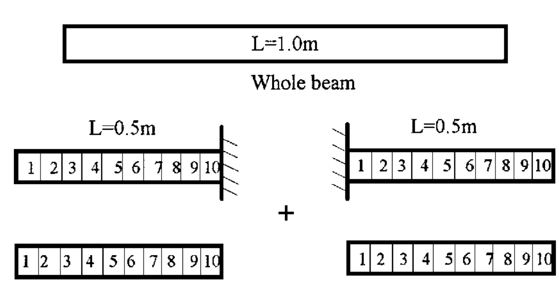

Fig. 1 Mixed modal synthesis for a beam with two substructures of equal length.

Table 2 Synthesis frequencies of free-free beam using proposed method, where ω (1/s) = $2\pi f$

	CBH		Craig		Mixed		Standard
i	f, Hz	Error, %	f, Hz	Error, %	f, Hz	Error, %	f, Hz
1, 2	0		0		0		0
3	11.2321	0.0000	11.2322	0.0009	11.2321	0.0000	11.2321
4	30.8707	0.0002	30.8706	0.0006	30.8707	0.0002	30.8708
5	60.2921	0.0039	60.2902	0.0007	60.2896	0.0003	60.2898
6	99.1976	0.0212	99.1968	0.0001	99.1965	0.0004	99.1969
7	147.408	0.0252	147.429	0.0394	147.370	0.0004	147.371
8[a]	204.571	0.0096	204.625	0.0362	204.551	0.0001	204.551
9	270.691	0.0878	273.037	0.9554	270.482	0.0107	270.453
10	344.954	0.0499	375.516	8.9140	346.865	0.6042	344.782
11	428.259	0.2390			499.562	5.2252	427.238

[a]Truncation frequency of synthesis modes (maximum error for the mixed method is less than 0.001%).

Table 3 Synthesis frequencies of free-free beam using proposed method, where ω (1/s) = $2\pi f$

	Mixed		Standard
i	f, Hz	Error, %	f, Hz
1, 2	0		0
3	11.2321	0.0002	11.2321
4[a]	30.8707	0.0002	30.8708
5	60.2896	0.0003	60.2898
6	99.1972	0.0003	99.1969
7	147.382	0.0075	147.371
8[b]	204.607	0.0276	204.551
9	271.253	0.2958	270.453
10	367.590	6.6153	344.782

[a]Truncation frequency of synthesis modes (maximum error for the mixed method is less than 0.0003%).
[b]Truncation frequency of synthesis modes (maximum error for the mixed method is less than 0.03%).

Table 4 Synthesis frequencies of free-free beam using proposed method, where ω (1/s) = $2\pi f$

	Mixed		Standard
i	f, Hz	Error, %	f, Hz
1, 2	0		0
3	11.2321	0.0002	11.2321
4[a]	30.8707	0.0002	30.8708
5	60.2896	0.0003	60.2898
6	99.1967	0.0002	99.1969
7	147.372	0.0005	147.371
8	204.554	0.0016	204.551
9	270.494	0.0150	270.453
10[b]	344.924	0.0412	344.782
11	428.827	0.3725	427.238
12	555.739	7.3831	517.529

[a]Truncation frequency of synthesis modes (maximum error for the mixed method is less than 0.0003%).
[b]Truncation frequency of synthesis modes (maximum error for the mixed method is less than 0.042%).

substructure was divided into 10 elements of equal length. FEM was used to calculate the mixed modes of the substructures, i.e., R rigid-body modes; the first L_{E1} and L_{E2} lowest free-free interfacial modes for the two substructures, respectively; the first L_{c1} lowest free-fixed (with the left-end free) interfacial modes for substructure 1; and the first L_{c2} lowest fixed-free (with the right-end free) for substructure 2.

Table 1 shows the truncation frequency of substructure modes $f_{EN} = f_{bN} = 123$ Hz ($L_{E1} = L_{E2} = 2$, $L_{c1} = L_{c2} = 3$), and Table 2 shows the truncation frequency of substructure modes $f_{EN} = f_{bN} = 237$ Hz ($L_{E1} = L_{E2} = 3$, $L_{c1} = L_{c2} = 4$). Tables 1 and 2 give the comparison of the frequencies of the free-free beam among the proposed mixed method (Mixed), CBH method[10,11] (CBH), Craig method[7] (Craig), and the reference standard solution (Standard). There is a reliable criterion for truncation frequency of system synthesis modes. When the system synthesis frequency f is lower than the truncation frequency $f_{EN} = f_{bN}$, the maximum error is less than 0.001%, and when system synthesis frequency f is higher than the truncation frequency $f_{EN} = f_{bN}$, the error increases quickly as synthesis frequency f increases.

Table 3 is for truncation frequency of substructure modes $f_{EN} = 235$ Hz and $f_{bN} = 44$ Hz ($L_{E1} = L_{E2} = 3$, $L_{c1} = L_{c2} = 2$), and Table 4 is for truncation frequency $f_{EN} = 381$ Hz and $f_{bN} = 44$ Hz ($L_{E1} = L_{E2} = 4$, $L_{c1} = L_{c2} = 2$) of substructure modes. Tables 3 and 4 give the comparison of the synthesis frequencies of the free-free beam between the proposed mixed method and the reference standard solution. There are two reliable criteria for truncation frequency of system synthesis modes. When the synthesis frequency f is lower than the truncation frequency f_{bN}, the maximum error is less than 0.0003%; when the synthesis frequency f is higher than the truncation frequency f_{bN} but lower than the truncation frequency f_{EN}, the maximum error is less than 0.05%; when the synthesis frequency f is higher than the truncation frequency f_{EN}, the error increases quickly as synthesis frequency f increases.

B. Example 2

Consider lateral vibration of a rectangular plate with two opposite sides clamped and the other two opposite sides free (Fig. 2). For this plate, length $2a = 4$ m, width $2b = 1$ m, and thickness $h = 0.04$ m. For the material of the plate, Young's modulus $E = 200 \times 10^9$ Pa, Poisson's ratio $\nu = 0.3$, and mass density $\rho = 7.8 \times 10^3$ kg/m^3. The calculated results based on FEM without reduction are used as the reference standard solutions for illustrating the computational accuracy of the proposed method.

The whole plate is divided into two substructures with length $a = 2$ m and width $2b = 1$ m (Fig. 2). Each substructure was divided into 4×2 equal rectangular plate elements. FEM was used to calculate the mixed modes of the substructures; they are the first L_{c1} and L_{c2} lowest fixed-fixed-interfacial modes for the two substructures; the first L_{E1} lowest fixed-free- (with the right-hand side free) interfacial modes for substructure 1, and the first L_{E2} lowest free-fixed- (with the left-hand side free) interfacial modes for substructure 2.

Table 5 is for truncation frequency of substructure modes $f_{EN} = f_{bN} = 299$ Hz ($L_{E1} = L_{E2} = 9$, $L_{c1} = L_{c2} = 6$). It gives a comparison of the synthesis frequencies of the rectangular plate with two opposite sides clamped and the other two opposite sides free among the proposed mixed method (Mixed), CBH method[10,11] (CBH), and the reference standard solution (Standard). Table 5 shows that there is a reliable criterion for truncation frequency of system synthesis modes. When the system synthesis frequency f is lower than the truncation frequency $f_{EN} = f_{bN} = 299$ Hz, the maximum error is less than 0.005%.

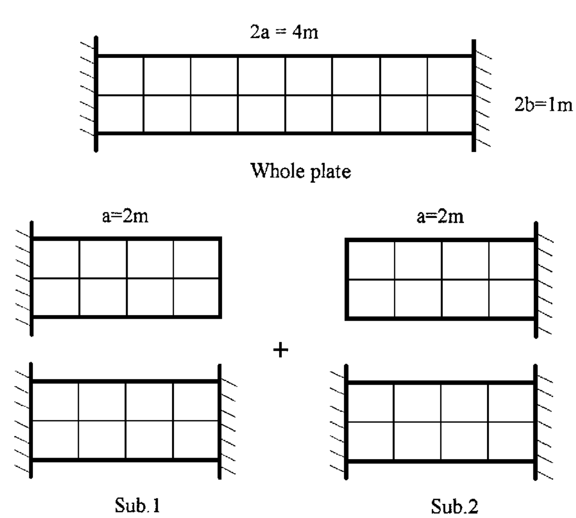

Fig. 2 Mixed modal synthesis for a plate with two substructures of equal length.

Table 5 Synthesis frequencies of fixed-fixed plate using proposed method, where ω (1/s) = $2\pi f$

	CBH		Mixed		Standard
i	f, Hz	Error, %	f, Hz	Error, %	f, Hz
1	13.436	0.000	13.436	0.000	13.436
2	36.306	0.003	36.305	0.000	36.305
3	37.911	0.003	37.910	0.000	37.910
4	71.961	0.003	71.959	0.000	71.959
5	79.201	0.000	79.201	0.000	79.201
6	117.18	0.009	117.17	0.000	117.17
7	128.80	0.016	128.78	0.000	128.78
8	170.79	0.018	170.76	0.000	170.76
9	187.66	0.011	187.64	0.000	187.64
10	229.01	0.052	228.90	0.004	228.89
11	231.67	0.035	231.60	0.004	231.59
12	252.12	0.024	252.07	0.004	252.06
13	262.05	0.015	262.02	0.004	262.01
14[a]	298.09	0.034	298.00	0.003	297.99

[a]Truncation frequency of synthesis modes (maximum error for the mixed method is less than 0.005%).

VI. Conclusions

The accurate expression of substructural displacements is given based on the mixed substructural modes, i.e., a mixed use of some prefound lower modes of the substructures with interfaces fixed or free. Such mixed modes are used instead of the quasicomparison functions before the Ritz procedure is executed. This leads to a new modal synthesis technique using mixed modes. This new method is a combination of the fixed-interfacial method, the free-interfacial method, and the assumed modes method using quasicomparison functions. The key point to realize this scheme is that the exact residual modes, which contain the effects of the higher free-interface modes, are expressed analytically in terms of some lower fixed-interface modes by means of the accurate expression, and the accurate substructural displacements are also expressed analytically in terms of some lower mixed modes. It has been proved that the truncation frequency f_{bN} of the substructural fixed-interface modes and the truncated frequency f_{EN} of the substructural free-interface modes are the two reliable criteria for truncation frequency of the system synthesis modes. These criteria indicate the accuracy and reliability of the synthesized results, and they are very important and useful in practical engineering and have great theoretical significance.

For both exact free-interface and exact fixed-interface modal synthesis techniques, nonlinear synthesis formulas are derived based only on lower modes involved in the synthesis process. Such nonlinear equations are then solved iteratively, and considerable computer time may be spent if the required accuracy is high. However, the method proposed needs only to solve linear synthesis equations, although the contributions of all higher modes have been included. As a result, the present method not only has a simple form, high precision, and efficiency but also provides two reliable criteria of truncation frequency of system synthesis modes; this has been demonstrated in detail by the numerical examples.

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References

[1]Meirovitch, L., and Kwak, M. K., "Convergence of the Classical Rayleigh–Ritz Method and Finite Element Method," *AIAA Journal*, Vol. 28, No. 8, 1990, pp. 1509–1516.

[2]Qiu, J. B., and Tan, Z. Y., "Residual Modes Substructure Synthesis Techniques," *Proceeding of the International Conference on Vibration Engineering* (Beijing, PRC), edited by Z. C. Zheng, International Academic, 1994, pp. 51–56.

[3]Ying, Z. G., Qiu, J. B., and Tan, Z. Y., "Exact Residual Modes and Their Synthesis Techniques," *Journal of Vibration Engineering*, Vol. 9, No. 1, 1995, pp. 38–46 (in Chinese).

[4]Hou, S. N., "Review of Modal Synthesis Techniques and a New Approach," *Shock and Vibration Bulletin*, No. 40, Pt. 4, 1968, pp. 25–39.

[5]MacNeal, R. H., "A Hybrid Method of Component Mode Synthesis," *Computers and Structures*, Vol. 1, No. 4, 1971, pp. 581–601.

[6]Rubin, S., "Improved Component-Mode Representation for Structural Dynamic Analysis," *AIAA Journal*, Vol. 13, No. 8, 1975, pp. 995–1006.

[7]Craig, R. R., and Chang, C. J., "Free-Interface Methods of Substructure Coupling for Dynamic Analysis," *AIAA Journal*, Vol. 14, No. 11, 1976, pp. 1633–1635.

[8]Wang, W. L., Du, Z. R., and Chen, K. Y., "Brief Commentary for Modal Synthesis Techniques and a New Improvement," *Acta Aeronautica et Astronautica Sinica*, No. 3, 1979, pp. 32–51 (in Chinese).

[9]Qiu, J. B., Ying, Z. G., and Williams, F. W., "Exact Modal Synthesis Techniques Using Residual Constraint Modes," *International Journal for Numerical Methods in Engineering*, Vol. 40, 1997, pp. 2475–2492.

[10]Craig, R. R., and Bampton, M. C. C., "Coupling of Substructures for Dynamic Analysis," *AIAA Journal*, Vol. 6, No. 7, 1968, pp. 1313–1319.

[11]Hurty, W. C., Du, Z. R., and Chen, K. Y., "Dynamic Analysis of Structural Systems Using Component Modes," *AIAA Journal*, Vol. 3, No. 4, 1965, pp. 678–685.

[12]Hu, H. C., "Natural Vibration for Multiple DOF Systems (Constraint Modes Synthesis Method)," *Acta Aeronautica et Astronautica Sinica*, No. 2, 1980, pp. 28–36 (in Chinese).

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